Direct measurement of current-phase relations in superconducting weak links*

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We report a method for the direct measurement of the current-phase \( i(\theta) \) relations in superconducting weak links. We show that by proper choice of experimental parameters, an accurate measurement of \( i(\theta) \) can be obtained in spite of obscuring effects of thermal fluctuations. This method has been used to obtain the current-phase relation of an oxidized niobium point contact and has shown that it conforms closely to the sinusoidal Josephson relation.

Recently several theoretical papers have indicated that the superconducting current-phase relation of many weak links should be distinctly non-sinusoidal with important higher-order harmonic components.\(^1\)\(^2\) Such departures from the simple Josephson current phase relation are expected to have profound effects on the operating characteristics of devices incorporating such weak links.\(^3\)\(^4\)\(^5\) However, unambiguous determinations of the current-phase relation \( i(\theta) \) have been exceptionally difficult to realize beyond the qualitative measurements of Fulton,\(^6\) Fulton and Dynes,\(^7\) and Song and Rochlin.\(^8\) This paper describes a method for direct quantitative measurements of \( i(\theta) \), analyzes the profound effect of thermal fluctuations on manifestations of \( i(\theta) \), and reports some results on the measured current-phase relation of an oxidized niobium point contact weak link.

When an external flux \( \phi_2 \) is applied to a superconducting ring of inductance \( L \) closed by the weak link to be studied, circulating supercurrents are induced and a flux \( \phi \) is admitted into the ring. A maximum supercurrent \( i_c \) can flow across the weak link. For the order parameter to be single valued, the phase difference \( \theta \) across the link must satisfy

\[
\theta = -2\pi(\phi/\phi_0), \quad \text{modulo } 2\pi, \quad (1)
\]

where \( \phi_0 \) is the flux quantum. The difference between \( \phi \) and \( \phi_0 \) yields the current in the link,

\[
i = (\phi - \phi_0)/L. \quad (2)
\]

Thus, Eqs. (1) and (2) show that by monitoring \( \phi \) as a function of \( \phi_0 \) we obtain a measure of both the current and the phase difference across the link. In the absence of fluctuations the current-phase relation \( i(\theta) \) could be determined from such measurements.

In practice only a time-average flux \( \langle \phi \rangle \) is measurable, where the averaging time, determined by the flux detector, is much longer than the equilibration time for \( \phi \). The flux \( \phi \) is disturbed by thermal fluctuations, so that in general \( \langle \phi \rangle \) does not equal \( \phi_\infty \), the value \( \phi \) would take in the absence of fluctuations. We measure an apparent current-phase relation defined by

\[
\bar{i}(\langle \theta \rangle) = \langle (\phi) - \phi_\infty \rangle/L, \quad (3)
\]

where

\[
\langle \theta \rangle = -2\pi\langle \phi \rangle/\phi_0, \quad (4)
\]

instead of the true current-phase relation

\[
i(\theta) = (\phi - \phi_\infty)/L. \quad (5)
\]

Because of fluctuations, \( \bar{i}(\langle \theta \rangle) \) differs from \( i(\theta) \), but we will show that if \( L_i \) is small enough \( \bar{i}(\langle \theta \rangle) \) is very close to \( i(\theta) \).

In order to calculate the fluctuation effect on \( \langle \phi \rangle \) and hence on the measured apparent current-phase relation, we model the weak link by a superconducting element with current-phase relation \( i(\theta) \) shunted by a capacitance \( C \) and a conductance \( \sigma_0 \). The equation of motion\(^9\) for this system is

\[
\frac{d}{d\phi} U(\phi, \phi_0) - C \dot{\phi}^2 + \sigma_0 \dot{\phi}, \quad (6)
\]

The potential \( U \), given by

\[
U(\phi, \phi_0) = (\phi - \phi_0)^2/2L + \int_0^\phi i(2\pi\phi'/\phi_0) d\phi', \quad (7)
\]

is the sum of a parabolic term and an oscillatory term (see Fig. 1). For large \( L_i \) there is a set \( \{\phi^{(m)}_\infty\} \) such that the local minima in \( U(\phi, \phi_0) \) occur when \( \phi = \phi^{(m)}_\infty \). This set defines the metastable states of \( \phi \). For small \( L_i \) there is only one minimum in \( U \) which occurs at \( \phi_\infty \). We define \( \gamma \) so that for \( L_i < \gamma \) there is only one minimum in \( U \) for all \( \phi_0 \) and for \( L_i > \gamma \) there is some \( \phi_0 \) for which \( U \) has at least two minima. For \( i(\theta) = i_c \sin\theta, \gamma = \phi_0/2\pi \); for arbitrary \( i(\theta) \), \( \gamma \) is always \( < \frac{1}{2} \phi_0 \). When \( L_i > \gamma \), accurate measurements of the current-phase relation cannot be made due to the effect of thermally activated transitions between the metastable states.

When \( L_i < \gamma \), the current-phase relation can be accurately mapped despite fluctuation effects. In
will have qualitative effects similar to intrinsic thermal fluctuations. If the amplitude and the spectrum of the external fluctuations are known, it is possible to reconstruct $\langle \phi \rangle$ by numerical techniques as it would appear without these fluctuations.

Our measurements are made by monitoring the flux inside the sample ring with an rf-biased SQUID flux detector weakly coupled to the sample through a normal-metal rf shield using a dc flux transformer. A periodic ramped flux (~0.1 Hz) of sufficient amplitude to induce several flux quanta is applied to the ring. For each cycle of applied flux, the flux in the ring is recorded in a 1024-channel signal averager with each channel corresponding to a value of $\phi_z$. This procedure eliminates low-frequency noise in the measuring system. Details of the apparatus have been previously described.\textsuperscript{10}

The weak coupling ~ 0.01 between the sample and the SQUID reduces magnetic noise transmitted to the sample through the flux transformer. SQUID sensitivity of $3 \times 10^{-6} \phi_0 / (Hz)^{1/2}$ provides ~2 $\times 10^{-5} \phi_0 / (Hz)^{1/2}$ resolution at the sample. Thus measurement of $\langle \phi \rangle$ to 0.01 $\phi_0$ accuracy for $10^9$ values of $\phi_z$ requires several hours of data acquisition. When measurement is completed, the contents of the signal-averager memory are transferred to a computer where $\bar{I}(\langle \theta \rangle)$ is derived and Fourier analyzed.

Equation (8) has been evaluated numerically for several values of $L$, $i_F$, and $T$ with $i(\theta)$ of the form $i_1 \sin \theta + i_2 \sin 2\theta$. In general the fluctuations result in $\langle \phi \rangle - \phi_{<}\ll \langle \phi_{\text{eq}} \rangle - \phi_{<}$. We find $\langle \phi \rangle$ departs most from $\phi_{\text{eq}}$ for those values of $\phi_{\text{eq}}$ where $i(\theta)$ has largest curvature. When $Li_c$ is only slightly less than $\gamma$, fluctuations have relatively greater effect than at smaller $Li_c$. Fluctuation effects are reduced as the parameter $\frac{\phi_0^2}{Li_c T}$ is increased. For example, consider $i(\theta) = 0.05(\phi_0 / L) \sin \theta$, i.e. $Li_c = 0.05\phi_0$. Expressing $\bar{I}(\langle \theta \rangle)$ as

\begin{align}
\bar{I}(\langle \theta \rangle) = \sum_n \tilde{I}_n \sin n\langle \theta \rangle
\end{align}

and computing the first terms in the expression, we find $\tilde{I}_1 = 0.041(\phi_0 / L)$, $\tilde{I}_2 = 0.009(\phi_0 / L)$ and $\tilde{I}_3 \leq 10^{-4}(\phi_0 / L)$. $\tilde{I}_3 \leq 10^{-4}(\phi_0 / L)$ when $\phi_0^2 / Li_c T = 100$, but $\tilde{I}_3 = 0.049(\phi_0 / L)$, $\tilde{I}_2 = 0.00021(\phi_0 / L)$, and $\tilde{I}_3 \leq 10^{-4}(\phi_0 / L)$ when $\phi_0^2 / Li_c T = 800$ (see Fig. 2). The most prominent fluctuation effect of $\bar{I}$ is the overall reduction in amplitude from $i$ although fluctuations also alter the harmonic content of $\bar{I}$. Despite fluctuation effects, it is possible to reconstruct $i(\theta)$ from $\bar{I}$ for any current-phase relation which is well represented by the first few terms of a Fourier series.

We also note that any fluctuation in $\phi_{<}$ which is faster than or comparable to the measuring time.
Measurements of \(i(\theta)\) were made on an air-oxidized niobium point contact closing a solid niobium ring. Typical results are shown in Fig. 3. The measured current-phase relation is nearly sinusoidal; we find \(i_1 = (0.054 \pm 0.001) \phi_0 / L, i_2 = (0.0015 \pm 0.0002) \phi_0 / L, \) and \(i_3 < 5 \times 10^{-4} \phi_0 / L\). These measurements were made with \(T = 6 \text{ K}\) and \(L = 3 \times 10^{-10} \text{ H}\) so that \(\phi_0^2 / L k_B T \approx 175\). For comparison, numerical evaluation of Eq. (8) at this same value of \(\phi_0^2 / L k_B T\) for \(i(\theta) = 0.061(\phi_0 / L) \sin \theta\) yields \(i_1 = 0.054(\phi_0 / L), i_2 = 0.0011(\phi_0 / L), \) and \(i_3 < 10^{-4}(\phi_0 / L)\). The excess in the measured \(i_3\) above the calculated value may be due to a small external flux noise at the sample, and thus \(i(\theta)\) for this sample may be exactly sinusoidal.

Some years ago Silver and Zimmerman\textsuperscript{12} (SZ) attempted to measure \(i(\theta)\) for a niobium point contact using a technique similar to ours. Their measured apparent current-phase relation was distinctly non-sinusoidal and was taken instead to be nearly triangular. Zimmerman\textsuperscript{13} later suggested that this result might have been due to thermal fluctuations, although he did not make a theoretical analysis. SZ studied rings with \(L_i = \frac{1}{3} \phi_0\) and \(L_i = \frac{1}{5} \phi_0\). The potential for a sinusoidal current-phase relation with \(L_i = \frac{1}{3} \phi_0\) is shown in Fig. 1(b). In this case there are two metastable states, \(\phi_{a1}^{(1)}\) and \(\phi_{a1}^{(2)}\), when \(\phi_1\) is near \(\frac{1}{3} \phi_0\). To go from \(\phi_{a1}^{(1)}\) to \(\phi_{a1}^{(2)}\) the flux must surmount a barrier \(\Delta U_{a12}\). For the reverse transition the barrier is \(\Delta U_{a21}\). Fluctuations drive the flux back and forth over the relatively low barriers. The lifetime of the state at \(\phi_{a1}^{(1)}\) is \(\tau = \tau^* e^{\Delta U_{a12} / k_B T}\). The usual case of high damping, \(1 / RC \gg (1 / L C)^{1/2}\), and \(\tau^*\) is the order of \(L / R\).\textsuperscript{14}

If \(T = 4 \text{ K}, L = 5 \times 10^{-10} \text{ H}, R = 0 \Omega, \) and \(\phi_0 = \frac{1}{2} \phi_0\) the two states are separated by a symmetric barrier \(\Delta U = 0.066 \phi_0^2 / L\). The lifetimes of the states are \(~10^{-4} \text{ sec}\). Thus measurements made over times longer than this lifetime yield a time-average flux \(\langle \phi \rangle = \frac{1}{2} \phi_0\), exactly half way between the two values of \(\phi_{a1}\). For \(\phi_0 \approx \frac{1}{2} \phi_0\), but close enough so that two metastable states exist, \(\langle \phi \rangle\) has a value between the two states which is dependent upon their relative lifetimes. Although slowly sweeping \(\phi_i\) back and forth produces a reversible curve, for some values of \(\phi_{a1}\), \(\langle \phi \rangle\) is far from \(\phi_{a1}\). We find that a purely sinusoidal \(i(\theta)\) yields \(i(\theta)\) very rich in harmonics. Thus, the triangular result of SZ is almost certainly due to the fluctuation effects just described and is consistent with a true sinusoidal current-phase relation.

(In the case where \(L_i \gg \gamma\) the barriers against reverse transitions become large, so that \(\phi\) is a hysteretic function of \(\phi_{a1}\). Transitions between states occur over a distribution in values of \(\phi_{a1}\). These distributions have been studied by Kurki-Järvi\textsuperscript{9} and Jackel et al.\textsuperscript{11})

In our analysis of the fluctuation effects we have implicitly assumed that \(i(\theta)\) is quasi-sinusoidal as is expected for a narrow weak link of length comparable to or shorter than the superconducting coherence length. For long weak links a proper treatment requires consideration of fluctuations in the superconducting pair density in the spirit of Langer and Ambegaokar.\textsuperscript{15} Our model of the weak link also assumes a conductance \(g_0 = 1 / R\) which is independent of \(\theta\). Recent studies\textsuperscript{16-18} have shown that an accurate model of some weak links requires the inclusion of a phase-dependent conductance so the total conductance \(\sigma = \sigma_0 + \sigma_1 \cos \theta\). An analysis of the system\textsuperscript{19} which includes the \(\sigma_1 \cos \theta\) term shows that the potential \(U\) is unaltered although the dynamics of the system are affected. For systems with \(\sigma_1\) comparable to \(\sigma_0\) we expect a modification of the metastable state lifetimes when \(L_i \gg \gamma\). However, when \(L_i < \gamma\) where \(i(\theta)\) is most readily measured, the only effect of \(\sigma_1 \cos \theta\) is a change in the equilibration time of \(\phi\). Since our measurements are made over times much greater than this modified equilibration time, Eq. (8) is unaffected and our results are independent of \(\sigma_1\).

We found that measurements of the flux in a superconducting ring closed by a weak link provide a powerful method for the study of weak link current-phase relations. Thermal fluctuations in the ring modify the internal flux, but the fluctuation effect is relatively small with \(L_i\) in the proper regime. We conclude that the current-phase relation of an oxidized niobium point contact is very nearly sinusoidal.

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